4th International Conference on
Combinatorics, Cryptography, Computer Science and Computing
November 20-21, 2019 - Iran University of Science & Technology

A remark on the connectivity of $\Gamma(M)$

Rezvan Varmazyar
Department of Mathematics
Khoy Branch, Islamic Azad University, Khoy 58168-44799, Iran
varmazyar@iaukhoy.ac.ir

ABSTRACT

Let R be a commutative ring and M be an R-module. The zero-divisor graph of M is defined as the graph $\Gamma(M)$ with the vertex set $Z({}_RM)=\{x\in M|(x:_R M)(y:_R M)M=0,\ 0\neq y\in M\}$ and two distinct vertices x, y of M are adjacent if and only if $(x:_R M)(y:_R M)M=0$. In this paper, the connectivity of $\Gamma(M)$ is investigated.

KEYWORDS: zero-divisor graph, minimum degree, connectivity, vertex-cut, multiplication module.

1 INTRODUCTION

Let R be a commutative ring with identity and Z(R) be its set of zero-divisors. The zero-divisor graph of the ring R, denoted by $\Gamma(R)$, is the simple graph associated to R such that its vertex set consists of all its nonzero zero divisors and that two distinct vertices x, y are joined by an edge if and only if $xy=0$. This concept is due to Beck [5], who let all the elements of R be vertices of $\Gamma(R)$ and was mainly interested in colourings. For the information about the zero-divisor graphs of commutative rings, see [2-3], [10-11] and [13]. Recently, assigning a graph to a module has received a good deal of attention from many authors, see for instance [6-7].

Let M be an R-module. Recall that an element x of M is called a zero-divisor, if x = 0 or $(x:_R M)(y:_R M)M=0$ for some nonzero y of M with $0\neq(y:_R M)\subset R$. The set of zero-divisors of M is denoted by $Z({}_RM)$. We associate (simple) graph $\Gamma(M)$ to M with vertices $Z({}_RM)\setminus\{0\}$, and two distinct vertices x and y are adjacent if and only if $(x:_R M)(y:_R M)M=0$.

A graph is said to be connected if for each pair of distinct vertices x and y, there is a finite sequence of distinct vertices $x=a_1,...,a_n=y$ such that each pair $\{a_i,a_{i+1}\}$ is an edge. Such a sequence is said to be a path, and the distance, d(x; y), between vertices x and y is the length of the shortest path connecting them (d(x, y) = 1 if there is no such path). The diameter of a connected graph G is

diam (G) = sup{ d(x, y) ; x and y are distinct vertices of G}.

A connected graph with more than one vertex has diameter 1 if and only if it is complete, that is, there exists an edge between each pair of distinct vertices. We denote the complete graph with n vertices by K_n. Also, the minimum degree of G, denoted by $\delta(G)$, is the minimum degree of its vertices. The set of vertices of G is denoted by V(G).

In [9] it was proved that $\Gamma(M)$ is connected. The concept of the connectivity of a zero-divisor graph can be found in [1]. The connectivity of a graph G, denoted by κ(G), is the smallest number of vertices in G whose deletion leaves either a disconnected graph or K_1. A vertex-cut is a subset $U\subseteq V(G)$ such that

G-U is disconnected. Therefore, $\kappa(G)$ is the size of smallest set U such that U is a vertex-cut of G or G-U has only one vertex. For any graph G, $\kappa(G) \le \delta(G)$. In this paper we extend the concepts of the connectivity to the zero-divisor graph Γ(M).

It was shown in many paper that $\Gamma(R)$ is a connected graph with diameter less than or equal to 3.

Theorem 1.1. Let R be a commutative ring. Then $\Gamma(R)$ is finite if and only if R is finite or an integral domain.

Proof: See Theorem 2.2 of [2].

In the following results the connectivity of $\Gamma(R)$ is investigated.

Theorem 1.2. Let R be a finite local ring then $\kappa(\Gamma(R)) = \delta(\Gamma(R))$.

Proof: see Theorem 3.1 of [1].

The following proposition gives a bound for $\kappa(\Gamma(R))$.

Proposition 1.3. Let R be a finite ring. Then $\kappa(\Gamma(R)) \ge (\frac{\delta(\Gamma(R))}{2})^{\frac{1}{3}} - \frac{1}{\sqrt{3}}$.

Proof: See Proposition 3.8 of [1].

Theorem 1.4. Let R be a commutative ring and p be a prime number. Then the connectivity of $\Gamma(Z_{p^2})$ is p−2.

Proof: see Theorem 3.1 of [11].

Theorem 1.5. Let R be a commutative ring and let p, q be a prime numbers. The connectivity of $\Gamma(Z_{p^m q^n})$ is min{p−1, q−1}.

Proof: see Theorem 3.4 of [11].

2 CONCLUSION

Recall that an R-module M is multiplication if N = (N : M)M for every submodule N of M. Clearly every ring is a multiplication module over itself and every cyclic R-module is a multiplication R-module. For more information about multiplication module see [4], [7], and [12].

Let $M = Z_n$ and R = Z. It is clear that Z_n is a multiplication Z-module.

Theorem 2.1. Assume that n is a positive integer greater than 1and n is not a prime number. Then the zero-divisor graph $\Gamma(Z_n)$ is simple if and only if n is square-free.

Proof: seeTheorem 5.1 of [8].

Corollary 2.2. The lattice diagram of the group $(Z_n,+)$ is obtained from the zero-divisor graph of the Z-module Z_n, where $n \geq 2$.
Proof: See Corollary 5.4 of [8].

Theorem 2.52. Let M be an R-module. Then $\Gamma(M)$ is a connected graph and

$$diam(\Gamma(M)) \leq 3.$$

Proof: See Theorem 2.5 of [8].

Theorem 2.2. Let p be a prime number and $M = Z_{p^n}$. If n≥3 then κ(Γ(M))=p-1.

Proof: It is clear that Z(M) = Rp and $0 = Rp^n \subset Rp^{n-1} \subset ... \subset Rp^2 \subset Rp$. By the proof of Theorem 4.1[9], we get κ(Γ(M))=p-1.

Theorem 2.3. Let $p_1 \prec p_2 \prec ... \prec p_n$ be distinct prime numbers and $M = Z_{p_1 \cdots p_n}$. Then

$$\kappa(\Gamma(M)) = p_1 - 1.$$

Proof: If n=2 then by the proof of Theorem 4.2[9], Γ(M) is a complete bipartite graph. So we get the result.
For $n \geq 3$, Γ(M) has $(p_1 - 1)(p_2 - 1)...(p_n - 1)$ complete subgraphs of order n. By the proof of same theorem, we get deg(p_1)= $p_1 - 1 = \delta(\Gamma(M))$, and there is no vertex-cut of cardinality less than $\delta(\Gamma(M))$, as needed.

Theorem 2.4. Let $p_1 \prec p_2 \prec ... \prec p_n$ be distinct prime numbers and Let $M = Z_{p_1} \oplus ... \oplus Z_{p_n}$.
Then $\kappa(\Gamma(M)) = p_1 - 1$.

Proof: Let $a_i \notin Z(Z_{p_n})$. Then $\deg(0, a_2, a_3, ..., a_n) = p_1 - 1 = \delta(\Gamma(M))$, and there is no vertex-cut of cardinality less than $\delta(\Gamma(M))$ in this graph. Hence the result is hold.

REFERENCES

[1] R. Akhtar, L. Lee, Connectivity of the zero-divisor graph for finit rings, preprint.

[2] D. F. Anderson, and P. S. Livingston,The zero-divisor graph of a commutativering, J. Algebra, 217(2) (1999), 434-447.

[3] D. F. Anderson,On the diameter and girth of a zero-divisor graph, II, HoustonJ. Math., 34(2) (2008), 361-371. press.

[4] A. Barnard, Multiplication modules, J. Algebra71(1981),174-178.

[5] I. Beck, Coloring of commutative rings, J. Algebra, 116(1) (1988), 208-226.

[6] M. Behboodi, Zero divisor graphs for modules over commutative rings, J. Commut. Algebra 4 (2012), no. 2, 175-197.

[7] Z. El-Bast, and P. F. Smith, Multiplication modules, Comm.Algebra16(1988),755-779.

[8] S. C. Lee and R. Varmazyar, Zero-divisor graphs of multiplication modules, Honam Math. J. 34 (2012), no. 4, 571-584.

[9] S. C. Lee and R. Varmazyar, Chromatic number of the zero-divisor graphs over modules, Commun. Korean Math. Soc. 34 (2019), No. 2, 391-399

[10] S. B. Mulay,Cycles and symmetries of zero-divisors, Comm. Algebra, 30(7)(2002), 3533-3558.

[11] B. S. Reddy, R. S. Jain, and N. Laxmikanth, Vertex and edge connectivity of the zero-divisor graph $\Gamma(Z_n)$, arXiv: 1807.02703v1, 2018.

[12] P. F. Smith, Some remarks on multiplication module, Arch. Math.50(1988),223-235.

[13] D. Weber,Various Topics on Graphical Structures Placed on Commutative Rings, PhD dissertation, The University of Tennessee, 2017.