4th International Conference on
Combinatorics, Cryptography, Computer Science and Computing
November 20-21, 2019 - Iran University of Science & Technology

A generalization of trades

Nasrin Soltankhah[1]

Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran

Abstract

A combinatorial trade is subset of a combinatorial configuration which may be ”exchanged” without changing overall parameters in the configuration. The combinatorial configuration may be a graph, a Latin square , or in the case of this talk, a block design.

Specifically, a trade is a subset of a block design which may be exchanged by a disjoint mate to obtain a new block design. The concept of trade and combinatorial trades in general can be generalized and here we introduce μ-way trades.

A μ-way (v, k, t) *trade* of volume m consists of μ disjoint collections T_1, $T_2, \ldots, T_\mu$ each of m blocks, such that for every t-subset of v-set V, the number of blocks containing this t-subset is the same in each T_i $(1 \leq i \leq \mu)$. In the other words any pair of T_i's is a (v, k, t) trade of volume m.

In this talk we will discuss this concept, present a number of new results and open problems.

Keywords: trade, volume, foundation

AMS Mathematical Subject Classification [2010]: 13D45, 39B42

1 Introduction and preliminaries

Given a set of v treatments V, let k and t be two positive integers such that $t < k < v$. A (v, k, t) trade $T = \{T_1, T_2\}$ of volume m consists of two disjoint collections T_1 and T_2, each one containing m k-subsets of V, called blocks, such that every t-subset of V is contained in the same number of blocks in T_1 and T_2. A (v, k, t) trade is called (v, k, t) Steiner trade if any t-subset of found(T) occurs in at most once in $T_1(T_2)$. A (v, k, t) trade is also a (v, k, t') trade, for all $0 < t' < t$. In a (v, k, t) trade, both collections of blocks must cover the same set of elements. This set of elements is called the foundation of the trade and is denoted by found(T). A (v, k, t) trade is called d-homogeneous if the number of occurences of each element of X in T_1 (T_2) is exactly d.

The concept of trade was first introduced by Hedayat (1960) in paper [6]. Hedayat and Li applied the method of trade-off and trades for building BIBDs with repeated blocks (1979-1980). Later, Steiner trades are used and renamed by Milici and Quattrocchi (1986) with the name of DMB (disjoint and mutually balanced). However, in 1916 Cole and Gumming used a concept that is $(v, 3, 2)$ trade of volume 4 and 6. Hwang in 1986 [4], Mahmoodian and Soltankhah [7], and Asgari and Soltankhah [2] deal with the existence

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and non-existence of (v, k, t) trades.

The μ-way (v, k, t) trade concept was defined recently in [8].

Definition 1.1. A μ-way (v, k, t) trade of volume m consists of μ disjoint collections T_1, T_2, $\cdots$, T_μ each of m blocks, such that for every t-subset of v-set V, the number of blocks containing this t-subset is the same in each T_i $(1 \leq i \leq \mu)$. In the other words any pair of (T_i, T_j), $i \neq j$ is a (v, k, t) trade of volume m.

Definition 1.2. A μ-way (v, k, t) trade is called μ-way (v, k, t) Steiner trade if any t-subset of found(T) occurs at most once in T_1 $(T_j, j \geq 2)$.

There exist various questions concerning μ-way trades. The most important of these questions are on the minimum volume, minimum foundation and the set of all possible volume sizes of μ-way trades. Not much is known for the mentioned questiones on μ-way (v, k, t) trades for $\mu \geq 3$ and most of the papers have been focused mainly on the case $\mu = 2$. Some results on the existence and non-existence of 3-way (v, k, t) trades can be found in [8, 9]. The minimum volume and minimum foundation size of (v, k, t) trade was obtained by Hwang [4] and recently the minimum volume and minimum foundation size of μ-way $(v, 3, 2)$ trades for each integer number $\mu \geq 3$ has been studied in [5]. Of course the μ-way trade was defined for μ-way trade including trade set and N-legged trade, see [3].

Trades have various applications in combinatorial design theory. For instance, in the problem related to structure of block designs, the method of constructing block designs, non-isomorphism block designs, block designs with repeated blocks, determining defining set and intersection problem in block designs. For more details, we explain the relation of trade with the intersection problem of block designs in following:
It is clear that if there exist three $t-(v, k, \lambda)$ designs (V, B) which intersect in the same set of m blocks, and which differ in the remaining blocks then we obtain a 3-way (v', k, t) trade of volume $b_v - m$ where $b_v = |B|$. Conversely let $D = (V, B)$ be a $t-(v, k, \lambda)$ design and $T = \{T_1, T_2, T_3\}$ be a 3-way (v, k, t) trade of volume m. If $T_1 \subset B$ we say that D contains the trade T, and if we replace $T_i (i = 2, 3)$ with T_1, then we obtain new designs $Di = (D \setminus T_1) \cup T_i$ which are denoted by $D_i = D + T_i$ with the same parameters of D, and $|D_i \cap D| = |D_i \cap D_j| = b_v - m$ for $2 \leq i, j \leq 3$. If there is not a 3-way (v, k, t) trade of volume m, then there does not exist three designs with intersection number $b_v \setminus m$. So the problem of determining the set of all possible volume sizes of a μ-way (v, k, t) trade is one of the most important problems in combinatorial subjects. This problem has been answered for $\mu = 2$ and in case $\mu = 3$ for some special values of k and t. But not much is known for the mentioned questiones about μ-way (v, k, t) trades for $\mu \geq 3$. Here, we present some results about the existence and non-existence of μ-way (v, k, t) trades of some special volumes for case $\mu = 3$ and $t = 3$ and finally we present some open problems.

2 Recursive constructions

In this section, we present some recursive constructions that will be used to construct trades of larger volumes.
Construction 1. [8] Let $T = \{T_1, T_2, \cdots, T_\mu\}$ be a μ-way (v, k, t) trade of volume m. Then based on T, a μ-way $(v + \mu, k + 1, t + 1)$ trade of volume μm can be constructed as follows:

T_1^*	T_2^*	$\cdots$	T_μ^*
x_1T_1	x_1T_2	$\cdots$	x_1T_μ
x_2T_2	x_2T_3	$\cdots$	x_2T_1
x_3T_3	x_3T_4	$\cdots$	x_3T_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x_\mu T_\mu$	$x_\mu T_1$	$\cdots$	$x_\mu T_{\mu-1}$

Example 2.1. Let $T = \{T_1, T_2, T_3\}$ be the following 3-way $(4, 2, 1)$ trade of volume 2.

T_1	T_2	T_3
12	13	14
34	24	23

Now, we construct $T^* = \{T_1^*, T_2^*, T_3^*\}$ of volume 6 by the method of Construction 2 as follows:

T_1	T_2	T_3
$x12$	$x13$	$x14$
$x34$	$x24$	$x23$
$y13$	$y14$	$y12$
$y24$	$y23$	$y34$
$z14$	$z12$	$z13$
$z23$	$z34$	$z24$

T^* is a 3-way $(7, 4, 3)$ trade of volume 6.

Remark 2.2. It has been shown in [8] that regardless isomorphism trade of volume 6 is unique.

Construction 2. Let $T = \{T_1, T_2, \cdots, T_\mu\}$ be a μ-way (v_1, k_1, t_1) trade of volume m_1 and $F = \{F_1, F_2, \cdots, F_\mu\}$ be a μ-way (v_2, k_2, t_2) trade of volume m_2 such that found$(T)\cap$found$(F) = \emptyset$. Then based on T and F, a μ-way $(v_1 + v_2, k_1 + k_2, t_1 + t_2 + 1)$ trade of volume $\mu m_1 m_2$ can be constructed as follows:

T_1^*	T_2^*	$\cdots$	T_μ^*
F_1T_1	F_1T_2	$\cdots$	F_1T_μ
F_2T_2	F_2T_3	$\cdots$	F_2T_1
F_3T_3	F_3T_4	$\cdots$	F_3T_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$F_\mu T_\mu$	$F_\mu T_1$	$\cdots$	$F_\mu T_{\mu-1}$

In which F_iT_j is the set concluding the union each block in F_i with each block in T_j.

Example 2.3. Let $T = \{T_1, T_2, T_3\}$ and $F = \{F_1, F_2, F_3\}$ be the following 3-way $(4, 2, 1)$ trades of volume 2.

T_1	T_2	T_3
12	13	14
34	24	23

F_1	F_2	F_3
x_1x_2	x_1x_3	x_1x_4
x_3x_4	x_2x_4	x_2x_3

Now, we construct $T^* = \{T_1^*, T_2^*, T_3^*\}$ of volume 12 by the method of Construction 2.

T_1	T_2	T_3
$x_1x_2 12$	$x_1x_2 13$	$x_1x_2 14$
$x_1x_2 34$	$x_1x_2 24$	$x_1x_2 23$
$x_3x_4 12$	$x_3x_4 13$	$x_3x_4 14$
$x_3x_4 34$	$x_3x_4 24$	$x_3x_4 23$
$x_1x_3 13$	$x_1x_3 14$	$x_1x_3 12$
$x_1x_3 24$	$x_1x_3 23$	$x_1x_3 34$
$x_2x_4 13$	$x_2x_4 14$	$x_2x_4 12$
$x_2x_4 24$	$x_2x_4 23$	$x_2x_4 34$
$x_1x_4 14$	$x_1x_4 12$	$x_1x_4 13$
$x_1x_4 23$	$x_1x_4 34$	$x_1x_4 24$
$x_2x_3 14$	$x_2x_3 12$	$x_2x_3 13$
$x_2x_3 23$	$x_2x_3 34$	$x_2x_3 24$

T^* is a 3-way $(8,4,3)$ trade of volume 12.

Remark 2.4. The 3-way trade of volume 12 is unique.

3 Results about the existence and non-existence 3-way $(v,k,3)$ trades

In this section, we obtain some results about the existence and non-existence 3-way $(v,k,3)$ trades of special volume sizes.

Theorem 3.1. *The minimum possible volume size of* 3*-way* $(v,k,3)$ *trade is* 12 *and the trade of this volume exists.*

Theorem 3.2. *A* 3*-way* $(v,4,3)$ *trade of volume* m*,* $12 < m < 18$ *does not exist.*

Theorem 3.3. *A* 3*-way (v,k,t) trade of volume* $m_0 = 2^{\lceil \frac{t}{2} \rceil} 3^{\lfloor \frac{t}{2} \rfloor}$ *exists for* $t \geq 1$.

Theorem 3.4. *A* 3*-way (v,k,t) trade of volume* $m_1 = 2^{\lceil \frac{t}{2} \rceil} 3^{\lfloor \frac{t}{2} \rfloor} + 2^{\lceil \frac{t-1}{2} \rceil} 3^{\lfloor \frac{t-1}{2} \rfloor}$ *exists for* $t \geq 2$.

4 Some open problems

At the end, some open problems can be discused.

1- What is the minimum volume for the existence of a 3-way (v,k,t) trade? Is it equals to $2^{\lceil \frac{t}{2} \rceil} 3^{\lfloor \frac{t}{2} \rfloor}$?
The above problem is true for $t = 2, 3$.

Let m_0 and m_1 be the positive integer numbers introduced in the two above theorems. Then the following problem can be arised.

2- Does there exist a 3-way (v,k,t) trade of volume m, $m_0 < m < m_1$?

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e-mail: soltan@alzahra.ac.ir, soltankhah.n@gmail.com