4th International Conference on
Combinatorics, Cryptography, Computer Science and Computing
November 20-21, 2019 - Iran University of Science & Technology

# Multi-objective Bi-matrix Game with Uncertain Payoffs

**Hamid Bigdeli***
*Institute for the Study of War, Army Command and Staff University*
*Tehran, I.R. Iran*
*E-mail: hamidbigdeli92@gmail.com*

**Javad Tayyebi**
*Department of Industrial Engineering, Faculty of Industrial and*
*Computer Engineering, Birjand University of Technology, Birjand, I.R. Iran*

**Hassan Hassanpour**
*Department of mathematical and statistics, Faculty of mathematics,*
*University of Birjand, Birjand, I.R. Iran.*

**ABSTRACT**

In this paper, uncertain multi-objective bi-matrix games are studied. The uncertainty theory is a new theory to consider the lack of information and knowledge about the parameters and variables of the problem. It has a major role in game theory and model behavior of human in real world. In this paper, we investigate the games which contain multiple objectives and all entries of their payoff matrices are uncertain. A new notion of expected efficient equilibrium points is introduced. Using the concept of expected values, the game is transformed to a crisp multi-objective model. By weighted sum approach, a single objective model is obtained. Then, a quadratic programming is introduced to obtain expected efficient equilibrium points. Finally, a numerical example is presented to illustrate the validity and applicability of the method.

**KEYWORDS:** Uncertainty Theory, Bi-matrix Game, Multi-objective Game, Expected efficient Equilibrium.

## 1 INTRODUCTION

Game theory is a branch of operations research to models mathematical behaviors governing conflict situations. It considers decision making in organizations whose outcomes depend on decisions of two or more autonomous players. Any player has not full control over outcomes. Game-theoretic approaches assume that these players will anticipate the opponents' moves, and act wisely. Two-person nonzero-sum games (bi-matrix games) are a special kind of non-cooperative games. These games can be expressed by a pair of payoff matrices. Interestingly, the two-person zero-sum games are as a special case of them.

In real-world decision-making problems, people want to attain simultaneous goals, that is, they have multiple objectives. Hence, it seems natural that the game theoretic approaches to conflict resolution require to handle multiple objectives simultaneously.

Players in real games have often incomplete information about the other players' (or even his own) payoffs. To model such games, some theories concerning uncertainty and vagueness are applied, like fuzzy theory, and grey theory. Uncertainty theory is another mathematical tool to model imprecise quantities of entities.

Uncertainty theory was founded by B. Liu [5] in 2007. The first fundamental concept in uncertainty theory is uncertain measure that is used to measure the belief degree of an uncertain event. The concepts of membership function and uncertain distribution are two basic tools to describe uncertain sets, whose membership function is intuitionistic for us but frangible for arithmetic operations. Fortunately, an uncertain distribution may be uniquely determined by a membership function. The concept of uncertain variables (neither random variables nor fuzzy variables) describes imprecise quantities in human systems.

Mula et.al. [6] studied matrix game under uncertain theory via entropy. They introduced the entropy function on strategies of the matrix game. Using uncertainty theory, they obtained a crisp linear programming problem depending upon the confidence level to solve it using genetic algorithm. Gao [4] considered bi-matrix games under uncertainty theory. He proposed three solution concepts of uncertain equilibrium strategies as well as their existence theorem.

In this paper, we address bi-matrix game in uncertain environment, in which the players have multiple objectives and their payoffs are uncertain variables. We introduce the concept of efficient equilibrium points for such games. Then, we present an approach to obtain them.

## 2 PRILIMINARIES

Let $\Gamma$ be a nonempty set, and $L$ a $\sigma$ -algebra over $\Gamma$. If $\Gamma$ is countable, usually $L$ is the power set of $\Gamma$. If $\Gamma$ is uncountable, for example $\Gamma = [0,1]$, usually $L$ is the Borel algebra of $\Gamma$. Each element in $L$ is called an event. Uncertain measure is a function from $L$ to $[0,1]$. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to each event $\Lambda \in L$ a number $M(\Lambda)$ which indicate the belief degree which $\Lambda$ will occur. In order to ensure that the number $M(\Lambda)$ has certain mathematical properties, Liu [5] proposed the following four axioms:

1) (Normality) $M(\Gamma) = 1$ for the universal set $\Gamma$.
2) (Monotonicity) $M(\Lambda_1) \le M(\Lambda_2)$ whenever $\Lambda_1 \subset \Lambda_2$ .
3) (Self-duality) $M(\Lambda) + M(\Lambda^c) = 1$ for any event $\Lambda$.
4) (Countable subadditivity) For every countable sequence of events $\{\Lambda_i\}$,

$$M\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

The set function $M$ is called an uncertain measure if it satisfies the mentioned properties.

The triple $(\Gamma, L, M)$ is called an uncertainty space. In order to obtain an uncertain measure of compound events, a product uncertain measure was defined in the following way.

Let $(\Gamma_k, L_k, M_k)$ be uncertainty spaces for $k = 1,2,...$, write $\Gamma = \Gamma_1 \times \Gamma_2 \times ...$ and $L = L_1 \times L_2 \times ...$ the product uncertain measure $M$ on the product $\sigma$-algebra L is satisfying

$$M\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\} = \min_k M_k\{\Lambda_k\}$$

An uncertain variable is a measurable function $\xi$ from an uncertainty space $(\Gamma, L, M)$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \,|\, \xi(\gamma) \in B\}$$

is an event.

The uncertainty distribution $\Phi : \mathbb{R} \to [0,1]$ of an uncertain variable $\xi$ is defined by $\Phi(x) = M\{\xi \le x\}$ for any real number $x$.

**Theorem**[5]**.** A function $\Phi : \mathbb{R} \to [0,1]$ is an uncertainty distribution if and only if it is an increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$.

**Definition.** An uncertain variable $\xi$ is called zigzag if it has an uncertainty distribution as follows

$$\Phi(x)=\begin{cases}0 & x\le a\\ \dfrac{x-a}{2(b-a)} & a\le x\le b\\ \dfrac{x+c-2b}{2(c-b)} & b\le x\le c\\ 1 & x\ge c\end{cases}$$

denoted by $Z(a,b,c)$ where $a$ , $b$ and $c$ are real numbers with $a<b<c$ .

An uncertain distribution $\Phi$ is said to be regular if its inverse function $\Phi^{-1}(\alpha)$ exists and it is unique for each $\alpha\in(0,1)$ . For zigzag uncertain variable $\Phi(x)$ , we have

$$\Phi^{-1}(\alpha)=\begin{cases}(1-2\alpha)a+2\alpha b & \alpha<0.5\\ (2-2\alpha)b+(2\alpha-1)c & \alpha\ge 0.5\end{cases}$$

**Definition.** The expected value of an uncertain variable $\xi$ is defined by

$$E[\xi]=\int_0^{\infty}M\{\xi\ge r\}dr-\int_{-\infty}^{0}M\{\xi\le r\}dr$$

provided that at least one of the two integrals is finite.

If $\xi$ is a regular variable with an uncertainty distribution *Φ*, then the expected value may be calculated by

$$E[\xi]=\int_0^{\infty}(1-\Phi(x))dx-\int_{-\infty}^{0}\Phi(x)dx$$

If $\xi\in Z(a,b,c)$ is a zigzag uncertain variable, then the expected value of $\xi$ is $E[\xi]=\dfrac{a+2b+c}{4}$ .

Let $\xi$ and $\eta$ be independent uncertain variables [5] with finite expected values. Then for any real numbers $a$ and $b$ , we have

$E[a\xi+b\eta]=aE[\xi]+bE[\eta]$ **.**

Thus, we have a method to rank uncertain variables as follows

$\xi\ge\eta$ if and only if $E[\xi]\ge E[\eta]$.

## 3 UNCERTAIN MULTI-OBJECTIVE BI-MATRIX GAME

Let $I=\{1,2,...,m\}$ and $J=\{1,2,...,n\}$ be pure strategies of players I and II, respectively. The mixed strategy of each player is a probability distribution on his pure strategy set. A mixed strategy describe a situation that the player, rather than choosing a particular pure strategy, will randomly select a pure strategy based on the given distribution. The mixed strategy spaces for players I and II are

$$X=\left\{(x_1,...,x_m)\in\Re^m\left|\sum_{i=1}^{m}x_i=1,\ x_i\ge 0\right.\right\},$$

and

$$Y=\left\{(y_1,...,y_m)\in\Re^n\left|\sum_{j=1}^{n}y_j=1,\ y_j\ge 0\right.\right\},$$

respectively.

We display bi-matrix game as $G(X,Y,A,B)$ , where $A=[\xi_{ij}],B=[\eta_{ij}]$ are $m\times n$ matrices, whose entries $\xi_{ij}$ and $\eta_{ij}$ represent the payoffs of players I and II associated with the strategy profile $(i,j)$ respectively.

In a real game, the decision environment is often characterized by a large number of possible strategies, complicated relations between strategy choices and their intricate influences to payoffs, thus making accurate or probabilistic estimation of the payoff matrices is impossible. For such situations, we may specify $\xi_{ij}$ as uncertain variable with uncertainty distribution $\Phi_{ij}$ , and $\eta_{ij}$ as uncertain variable with uncertainty distribution $\psi_{ij}$ , for all $i = 1,2,...,m$ and $j = 1,2,...n$ .

In the following, it is assumed that all the payoffs are regular uncertain variables, for example, zigzag uncertain variables.

In real world decision making problems facing humans today, people want to attain simultaneous goals, that is, they have multiple objectives. Assume that each player has $p$ objectives.

Denote the uncertain payoff matrices of player I by $\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_P$ and, uncertain payoff matrices of player II as $\tilde{B}_1,\tilde{B}_2,...,\tilde{B}_P$ .

When player I chooses a mixed strategy $x \in X$ and player II choose $y \in Y$ , the expected payoff vectors of players are the expressed as follows:

$$x^T \tilde{A} y = (x^T \tilde{A}_1 y, x^T \tilde{A}_2 y, ..., x^T \tilde{A}_p y)$$

$$x^T \tilde{B} y = (x^T \tilde{B}_1 y, x^T \tilde{B}_2 y, ..., x^T \tilde{B}_p y)$$

where $\tilde{A} = (\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_p)$ and $\tilde{B} = (\tilde{B}_1,\tilde{B}_2,...,\tilde{B}_p)$ .

Now, we should express the definition of equilibrium strategies in this game. In this paper, we use expected value criterion to characterize the behaviors of players.

**Definition.** In an uncertain multi-objective bi-matrix game, a pair of strategies $(x^*, y^*) \in X \times Y$ is said to be expected efficient equilibrium solution if there does not exist another $(x, y) \in X \times Y$ such that

$$E\left(x^* \tilde{A}_k Y^*\right) \le E\left(x \tilde{A}_k Y^*\right) \quad k = 1,...,p$$

$$E\left(x^* \tilde{B}_k Y^*\right) \le E\left(x^* \tilde{B}_k Y\right) \quad k = 1,...,p$$

and

$$E\left(x^* \tilde{A}_l Y^*\right) < E\left(x \tilde{A}_l Y^*\right),$$

$$E\left(x^* \tilde{B}_t Y^*\right) < E\left(x \tilde{B}_t Y^*\right).$$

for at least two indices l, t.

From multi-objective optimization viewpoint, the strategy profile $(x^*, y^*) \in X \times Y$ is an expected efficient equilibrium point if and only if $x^*$ and $y^*$ are respectively efficient solutions for two following optimization problems

$$\max \left(E(x^T \tilde{A}_1 y^*),...,E(x^T \tilde{A}_p y^*)\right) \quad x \in X \tag{1}$$

$$\max \left(E(x^{*T} \tilde{B}_1 y),...,E(x^{*T} \tilde{B}_p y)\right) \quad y \in Y \tag{2}$$

A common method to solve multi-objective programming model is to reduce it to a single objective programming model.

Assume that players present weighting coefficient vector

$$\lambda = (\lambda_1,...,\lambda_p) \in \left\{ \lambda \in \Re^p \,\middle|\, \lambda_k \ge 0, k = 1,...,p, \sum_{k=1}^{p} \lambda_k = 1 \right\},$$

and

$$\gamma = (\gamma_1,...,\gamma_p) \in \left\{ \lambda \in \Re^p \,\middle|\, \gamma_k > 0, k = 1,...,p, \sum_{k=1}^{p} \gamma_k = 1 \right\},$$

respectively.

Using the weighting method, the problems (1) and (2), are written as follows:

$$\max \left( E\,(x^T \sum_{k=1}^{p} \lambda_k \tilde{A}_k y^*) \right) \quad (3)$$

$$x \in X,$$

$$\max \left( E\,(x^{*T} \sum_{k=1}^{p} \gamma_k \tilde{A}_k y) \right) \quad (4)$$

$$y \in Y.$$

Now, we consider an existence theorem which ensure the significance of our new concepts.

**Theorem.** Let $\xi_{ij}^k$ and $\eta_{ij}^k$ in the payoff matrices $\tilde{A}^k$ and $\tilde{B}^k$ be independent regular uncertain variables for $i = 1,\dots,m$ and $j = 1,\dots,n$. Then there exists at least one expected efficient equilibrium, say $(x^*, y^*)$ and the expected value of the uncertain bi-matrix game is $\left(x^* \tilde{A}_{Ew} y^*, x^* \tilde{B}_{Ew} y^*\right)$ where $\tilde{A}_{Ew} = \left( E[\sum_{k=1}^{p} \lambda_k \xi_{ij}^k] \right)_{m\times n}$, $\tilde{B}_{Ew} = \left( E[\sum_{k=1}^{p} \gamma_k \eta_{ij}^k] \right)_{m\times n}$.

**Proof:**

According to expected value criterion, and since there exists at least one equilibrium solution in a single-objective bi-matrix game, it is known that there also exists at least one efficient equilibrium solution.

Similar to crisp bi-matrix games, in order to find uncertain equilibrium point in an uncertain bi-matrix game, we present a sufficient and necessary condition in the following theorem.

**Theorem.** A strategy profile $(x^*, y^*)$ is an expected efficient equilibrium if and only if the point $(x^*, y^*, u^*, v^*)$ is an optimal solution to the following quadratic programming model

$$\begin{aligned} &\max_{x,y,u,v} \; x^T \left(\tilde{A}_{Ew} + \tilde{B}_{Ew}\right) y - u - v \\ &\tilde{A}_{Ew}\, y \le u e_m \\ &\tilde{B}_{Ew}\, x \le v e_n \\ &x \in X,\; y \in y \end{aligned}$$

where $u^* = x^{*T} \tilde{A}_{Ew} y^*$ and $v^* = x^{*T} \tilde{B}_{Ew} y^*$. $e_m$ and $e_n$ are, respectively, m- and n-dimensional column vectors whose elements are all ones.

## 4 NUMERICAL EXAMPLE

Suppose each of the players I and II have two objectives and two strategies such that payoff matrices are as follows

$$\tilde{A}_1 = \begin{bmatrix} Z(20,25,30) & Z(30,50,55) \\ Z(50,55,90) & Z(40,50,55) \end{bmatrix},$$

$$\tilde{A}_2 = \begin{bmatrix} Z(70,75,80) & Z(80,110,115) \\ Z(80,85,100) & Z(20,60,65) \end{bmatrix},$$

and

$$\tilde{B}_1 = \begin{bmatrix} Z(30,40,45) & Z(40,50,60) \\ Z(40,45,60) & Z(20,30,50) \end{bmatrix},$$

$$\tilde{B}_2 = \begin{bmatrix} Z(50,65,75) & Z(80,100,120) \\ Z(70,80,90) & Z(50,80,100) \end{bmatrix},$$

where entries of matrices are zigzag uncertain values.

Assume that the players present the same weights to two objectives ($\lambda_1 = \lambda_2 = \gamma_1 = \gamma_2 = \frac{1}{2}$). Using the expected value of uncertain variable, weighted expected matrices are

$$\tilde{A}_{Ew} = \begin{bmatrix} 50 & 75 \\ 75 & 50 \end{bmatrix},$$

$$\tilde{B}_{Ew} = \begin{bmatrix} 46.25 & 75 \\ 63.75 & 55 \end{bmatrix}$$

Thus, quadratic programming model is

$$\max \ (x_1, x_2) \begin{bmatrix} 96.25 & 150 \\ 138.75 & 105 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - u - v$$

$$\begin{bmatrix} 50 & 75 \\ 75 & 50 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} u \\ u \end{pmatrix} \le 0$$

$$\begin{bmatrix} 46.25 & 75 \\ 63.75 & 55 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} v \\ v \end{pmatrix} \le 0$$

$$x_1 + x_2 = 1$$

$$y_1 + y_2 = 1$$

$$x_1, x_2 \ge 0, \ y_1, y_2 \ge 0$$

Solving the model by Lingo software, the optimal solution is

$$\left(x_1^*, x_2^*, y_1^*, y_2^*, u^*, v^*\right) = \left(0.23, 0.77, 0.5, 0.5, 62.5, 59.7\right)$$

Therefore, the expected efficient equilibrium is

$$x^* = (0.23, 0.77),$$

$$y^* = (0.5, 0.5).$$

The objective values of players obtain as follows:

$$x * \tilde{A}_{1E} Y * = 51.03,$$

$$x * \tilde{A}_{2E} Y * = 73.97,$$

$$x * \tilde{B}_{1E} Y * = 41,$$

$$x * \tilde{B}_{2E} Y * = 79.49.$$

## 5 CONCLUSION

In this paper, multi-objective bi-matrix games with uncertain payoffs has been considered. In real world, there are situations in which payoffs are not exactly known. One of the methods to model these data is using uncertain variables. We presented a solution method to model bi-matrix game with multiple objectives in which the payoffs are uncertain variables. A numerical example demonstrates the validity and applicability of method to uncertain multi-objective bi-matrix game.

## REFERENCES


[1] Bigdeli H., Hassanpour H., Tayyebi J. “Constrained Bimatrix Games with Fuzzy Goals and its Application in Nuclear Negotiations”; Iranian Journal of Numerical Analysis and Optimization , 2018, 8, 81-110.

[2] Bigdeli, H.; Hassanpour, H.; Tayyebi, J. “Multiobjective Security Game with Fuzzy Payoffs”; IJFS; 2018, ():-doi:10.22111/ijfs.2018.3902.

[3] Bigdeli, H., Hassanpour, H. and Tayyebi, J. *A satisfactory strategy of multiobjective two person matrix games with fuzzy payoffs*, Iranian Journal of Fuzzy Systems Vol. 13, No. 4, (2016) pp. 17-33.

[4] Gao, J. (2007). Uncertain bimatrix game with applications. Fuzzy Optim Decis Making, DOI 10.1007/s10700-012-9145-6.

[5] Liu, B. (2007). Uncertainty theory (2nd ed.). Berlin: Springer.

[6] Mula, P., Sankar, K. R., Chandan, B., Matrix Game Under Uncertainty Theory via Entropy, International Journal of Innovative Science, Engineering & Technology, Vol. 3 Issue 2, February 2016.